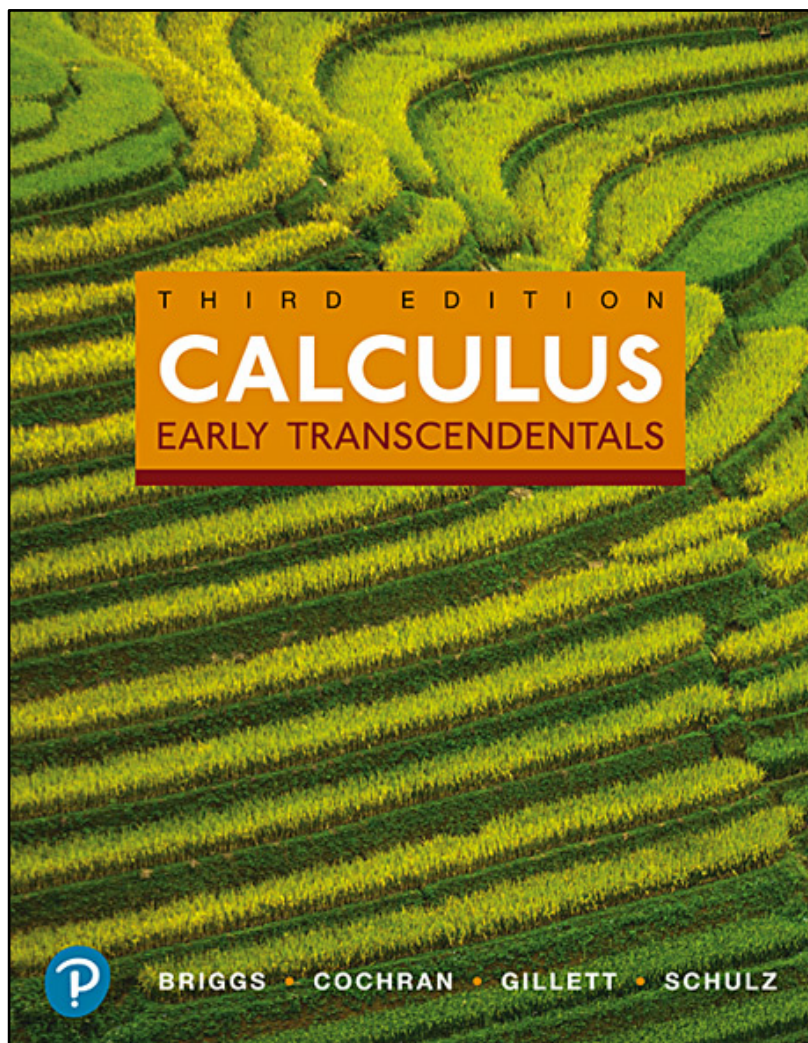


Calculus Early Transcendentals

Third Edition



Chapter 5 Integration

5 lectures

Section 5.1 Approximating Areas under Curves

HW # 2, 7, 8, 9, 11, 13, 15, 17, 23, 24, 27, 47, 49, 71, 73

- what is the area of the region R?

Figure 5.6

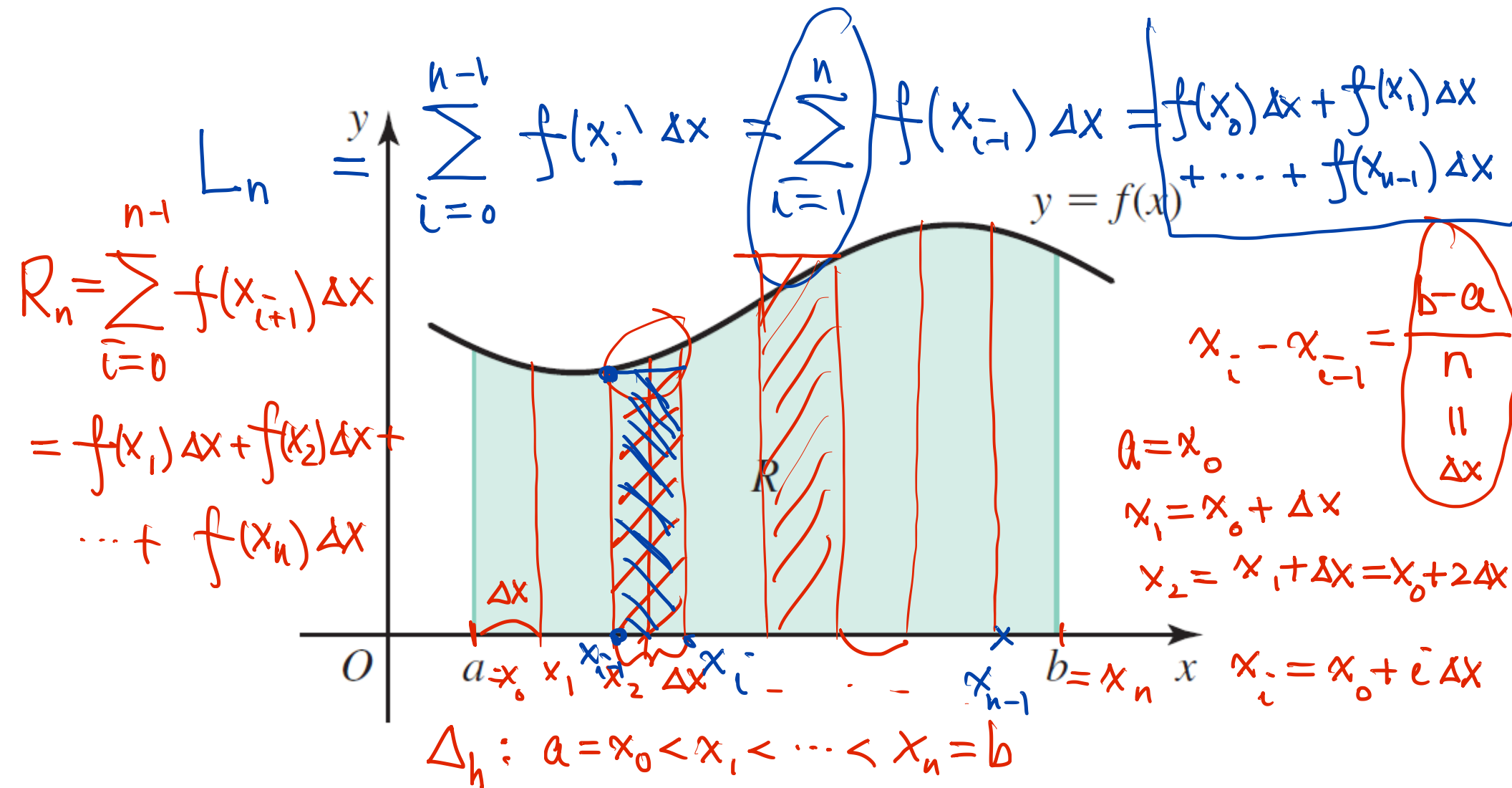
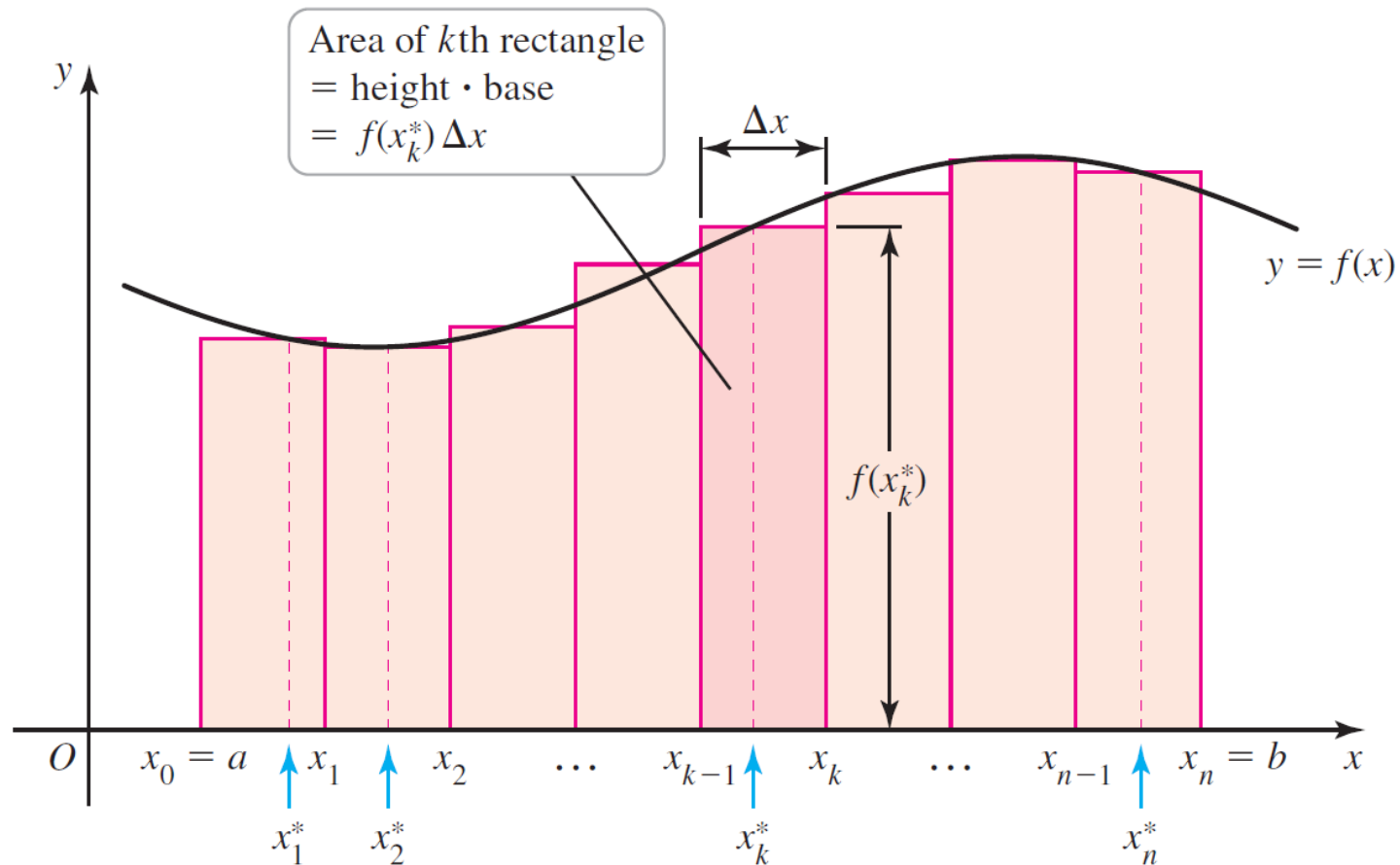
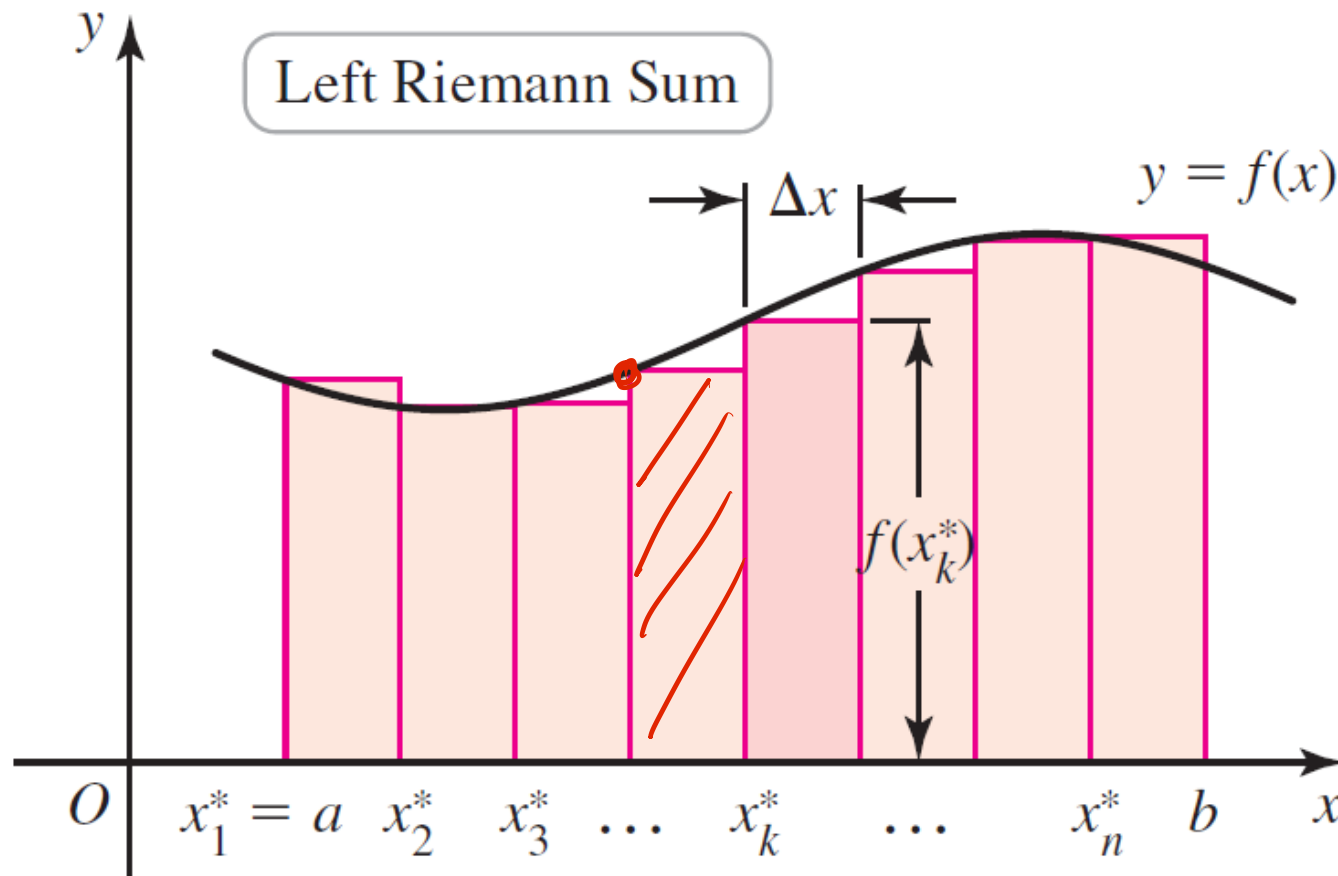


Figure 5.8



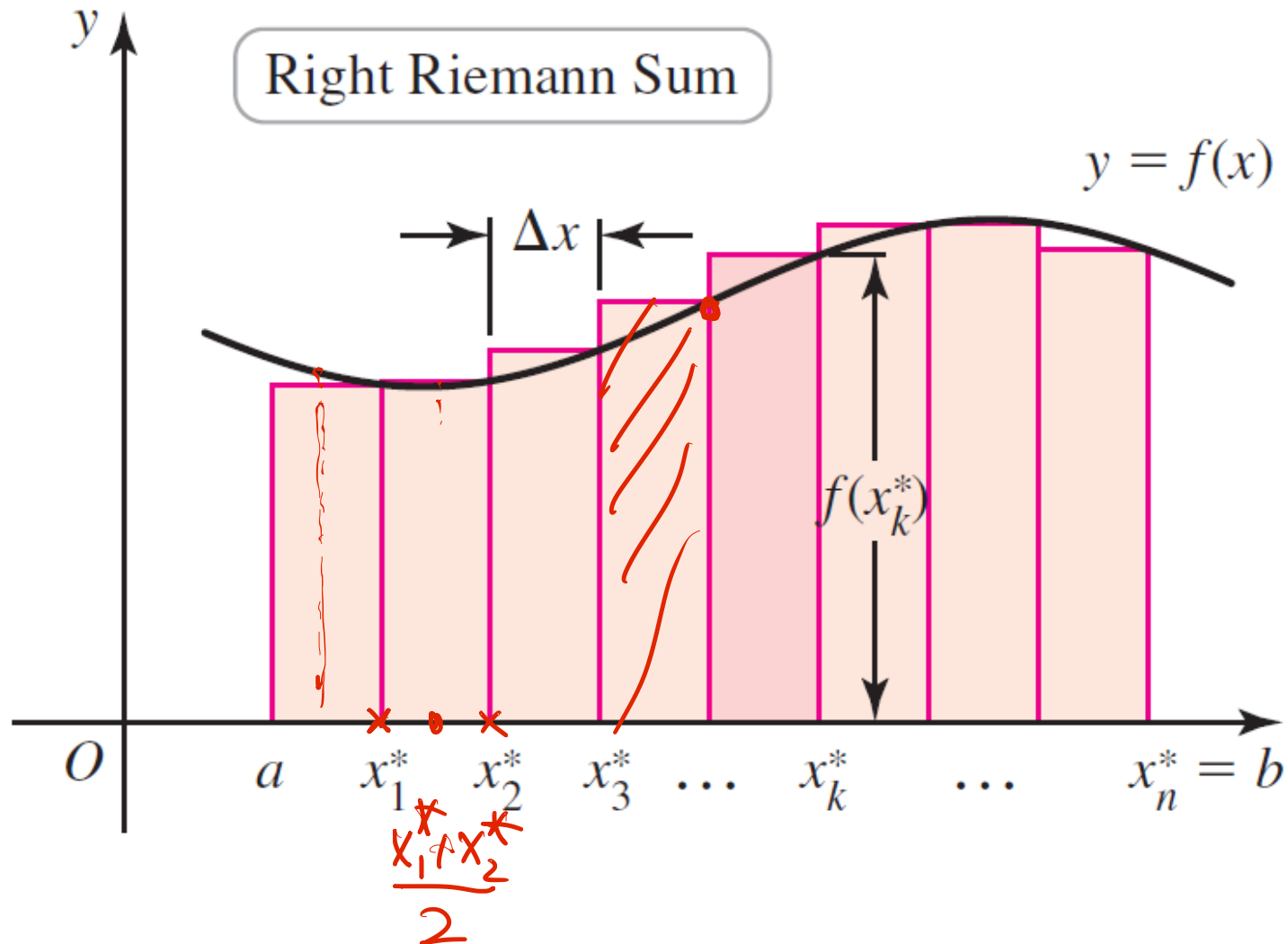
$$R_n^l =$$

Figure 5.9



$$R_n^r =$$

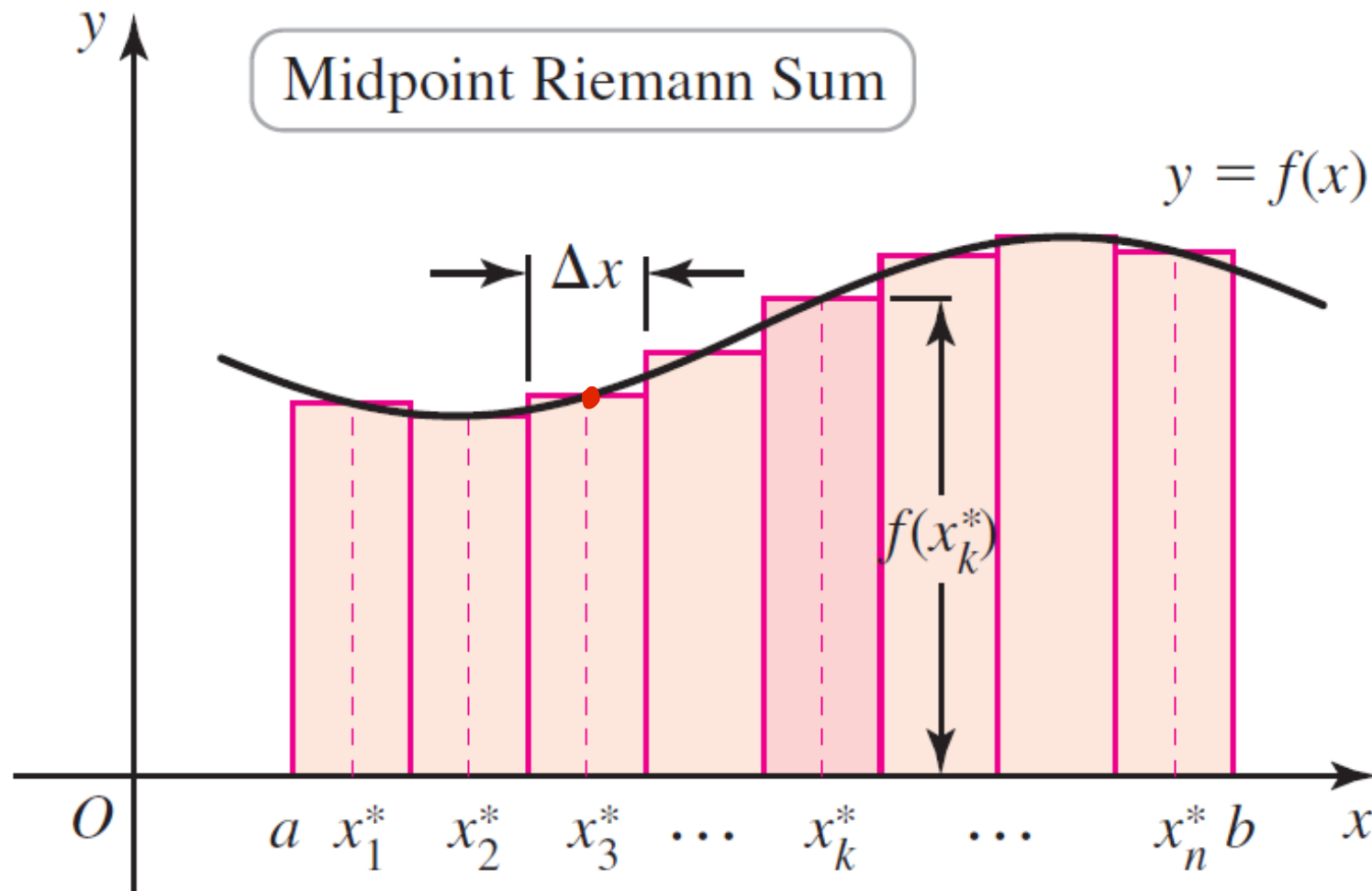
Figure 5.10



$$R_n^m = f\left(\frac{x_0^* + x_1^*}{2}\right) \Delta x + f\left(\frac{x_1^* + x_2^*}{2}\right) \Delta x + \cdots + f\left(\frac{x_{n-1}^* + x_n^*}{2}\right) \Delta x$$

$\parallel$
 m_n

Figure 5.11



(1) $f(x) = x$ on $[0, 1]$

partition $\Delta_n: 0 = x_0 < x_1 < \dots < x_n = 1$

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$x_i = 0 + i \Delta x = \frac{i}{n}$$

Riemann sum

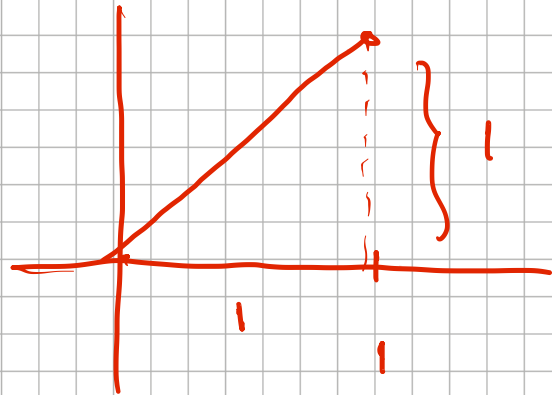
$$R_n = f(x_0) \Delta x + f(x_1) \Delta x + \dots + f(x_{n-1}) \Delta x$$

$$= \left[f(0) + f\left(\frac{1}{n}\right) + \dots + f\left(\frac{n-1}{n}\right) \right] \Delta x$$

$$= \left[0 + \frac{1}{n} + \frac{2}{n} + \dots + \frac{n-1}{n} \right] \cdot \frac{1}{n}$$

$$= \left(\frac{1}{n}\right)^2 [1 + 2 + \dots + (n-1)] = \left(\frac{1}{n}\right)^2 \frac{[1 + (n-1)] \cdot (n-1)}{2} = \frac{1}{n^2} \cdot \frac{n(n-1)}{2}$$

$$= \frac{n-1}{2n} \rightarrow \frac{1}{2}$$



(2) $f(x) = x^2$ on $[0, 1]$

partition $\Delta_n:$

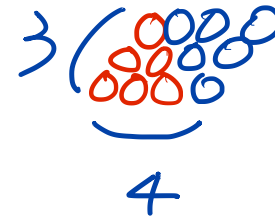
Riemann sum

Theorem 5.1 Sums of Powers of Integers

Let n be a positive integer and c a real number.

$$\sum_{k=1}^n c = cn$$

$1+2+\cdots+n$



$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

Example 5 $f(x) = x^3 + 1$ on $[a, b] = [0, 2]$ $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $x_i = a + i\Delta x = \frac{2i}{n}$

$$R_n^r = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = \left[\left(\frac{2}{n}\right)^3 + 1 + \left(\frac{2 \cdot 2}{n}\right)^3 + 1 + \dots + \left(\frac{2 \cdot n}{n}\right)^3 + 1 \right] \Delta x$$

Figure 5.15

$$= \frac{\left(\frac{2}{n}\right)^3}{n} \left[1 + 2^3 + \dots + n^3 \right] + n \cdot \frac{2}{n}$$

$$= \frac{\left(\frac{2}{n}\right)^3}{n} \left[\sum_{k=1}^n k^3 + n \right] + 2$$

$$= \frac{\left(\frac{2}{n}\right)^3}{n} \left[\frac{n^2(n+1)^2}{4} + n \right] + 2$$

$$= \frac{2^4 n^2(n+1)^2}{4 n^4} + n \cdot \frac{\left(\frac{2}{n}\right)^3}{n} + 2$$

$$= 4 \left(1 + \frac{1}{n}\right)^2 + \frac{2^4}{n^3} + 2$$

$$\rightarrow 4 + 2 = 6$$

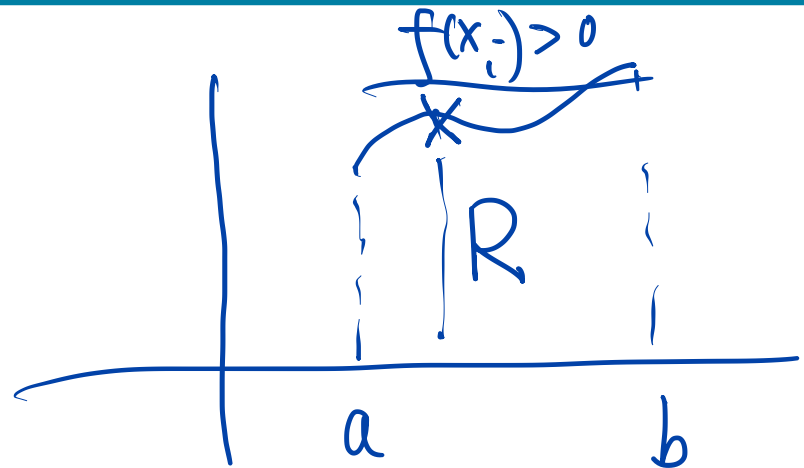
$f(x) = x^3 + 1$

Table 5.3: Left, Right, and Midpoint Riemann Sum Approximations

n	L_n	R_n	M_n
20	5.61	6.41	5.995
40	5.8025	6.2025	5.99875
60	5.86778	6.13444	5.99944
80	5.90063	6.10063	5.99969
100	5.9204	6.0804	5.9998
120	5.93361	6.06694	5.99986
140	5.94306	6.05735	5.9999
160	5.95016	6.05016	5.99992
180	5.95568	6.04457	5.99994
200	5.9601	6.0401	5.99995

Section 5.2 Definite Integrals

$$A(R) = \int_a^b f(x) dx = \lim_{\Delta x \rightarrow 0} R_n$$



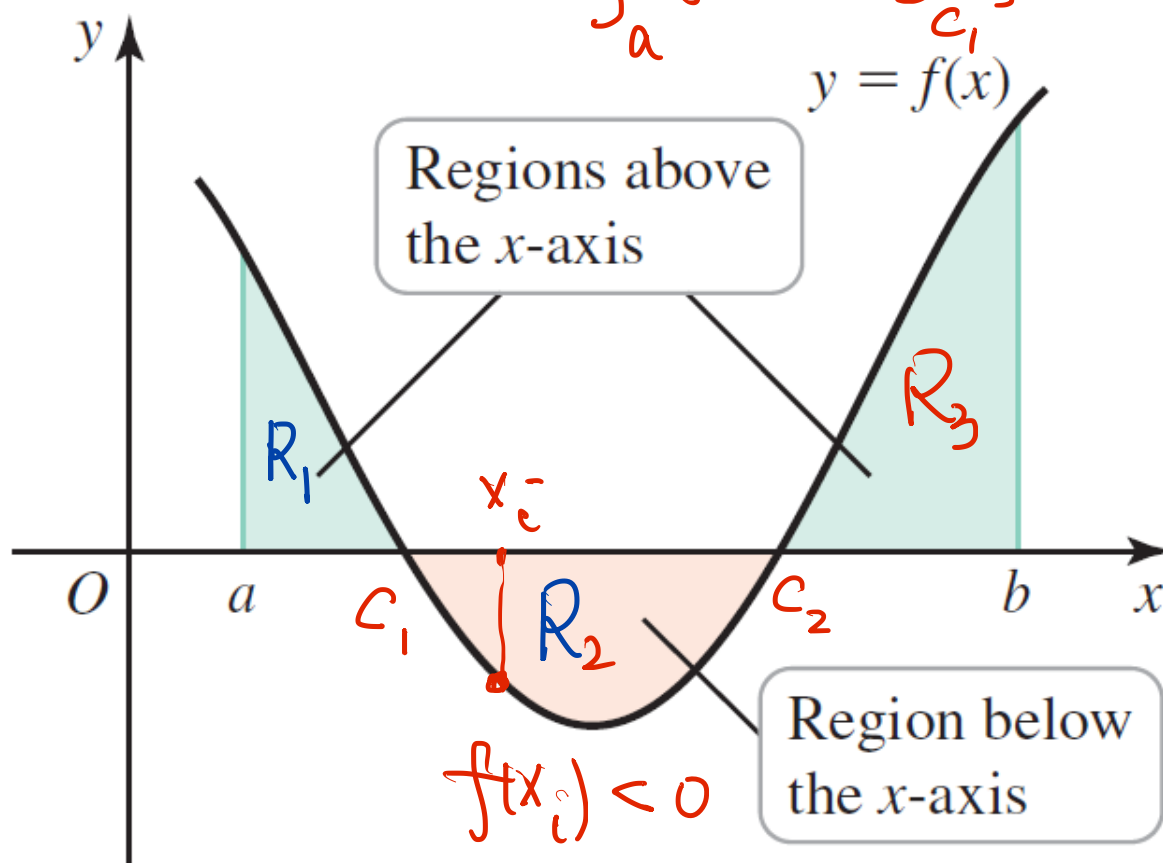
$$\int_a^b f(x) dx = A^+ - A^-$$

Figure 5.19

$$A(R) = A(R_1) + A(R_2) + A(R_3)$$

$$= \int_a^{c_1} f(x) dx - \int_{c_1}^{c_2} f(x) dx + \int_{c_2}^b f(x) dx$$

$$= \int_a^{c_1} f(x) dx + \int_{c_1}^{c_2} -f(x) dx + \int_{c_2}^b f(x) dx$$



$$f(x_i) < 0$$

$$-f(x_i) \Delta x < 0$$

Figure 5.21

Upper limit of integration

Upper limit of summation

Lower limit of integration

Lower limit of summation

Integrand

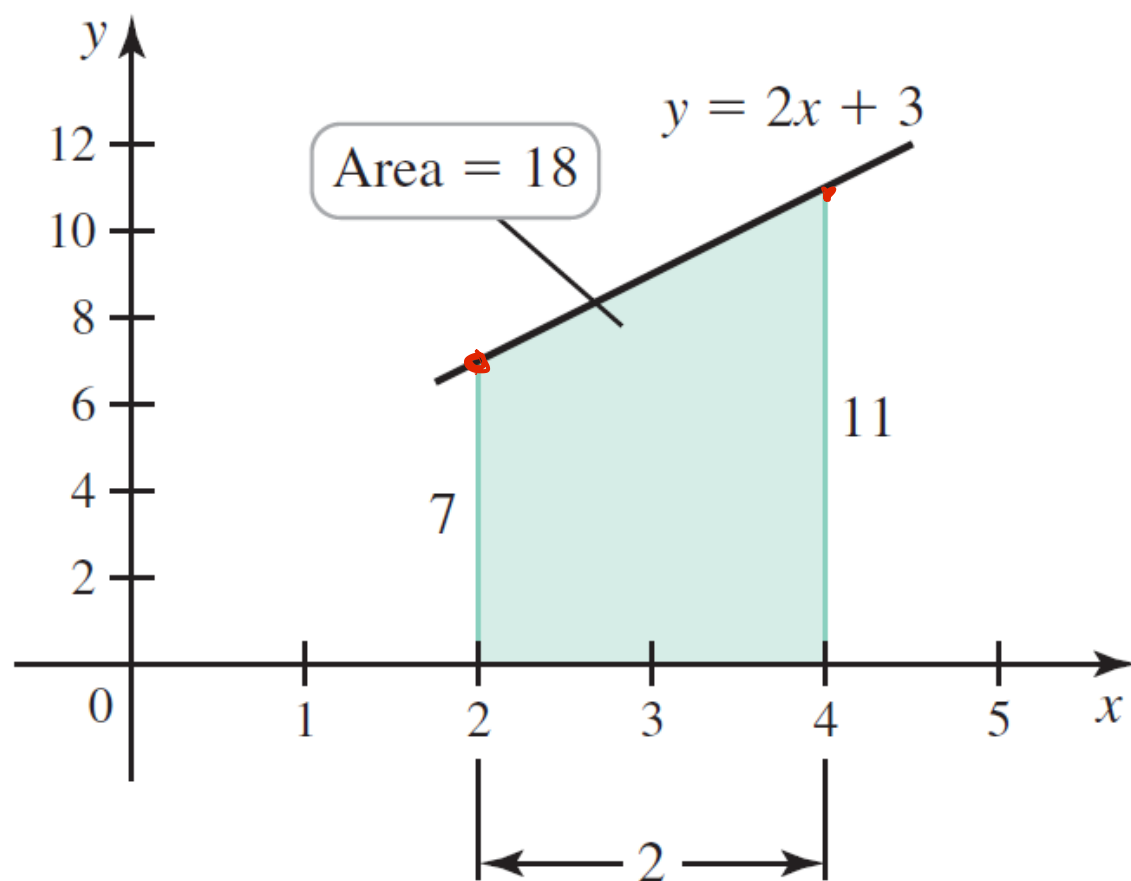
x is the variable of integration.

$$\int_a^b f(y) dy = \int_a^b f(x) dx = \lim_{\Delta \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

- Evaluate definite integrals using geometry

$$(a) \int_2^4 (2x + 3) dx = \frac{7+11}{2} \cdot 2 = 18$$

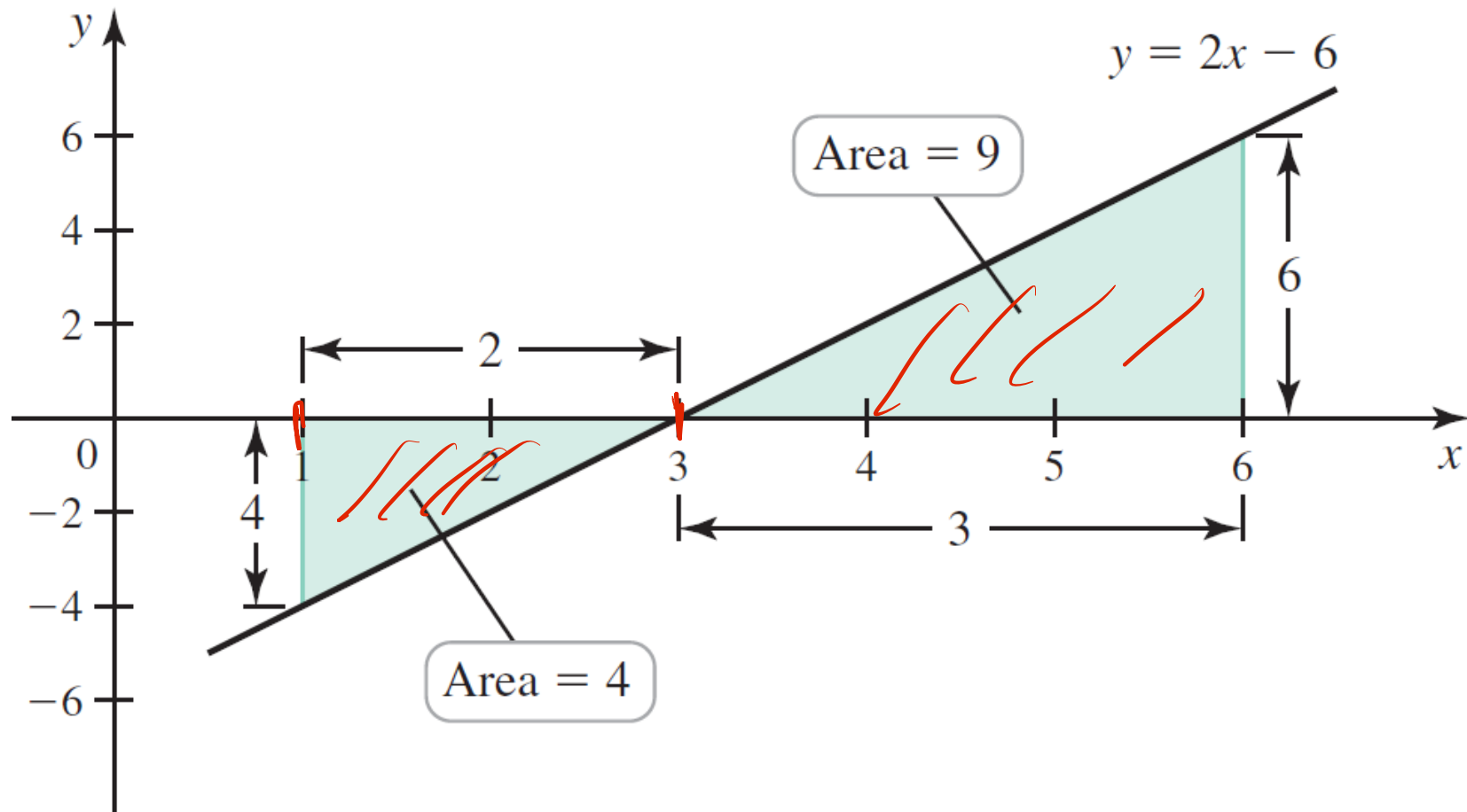
Figure 5.25



$$(b) \int_1^6 (2x-6) dx = -\frac{1}{2} 4 \cdot 2 + \frac{1}{2} 3 \cdot 6$$

$$= -4 + 9 = 5$$

Figure 5.26



$$(c) \int_3^4 \sqrt{1 - (x-3)^2} dx$$

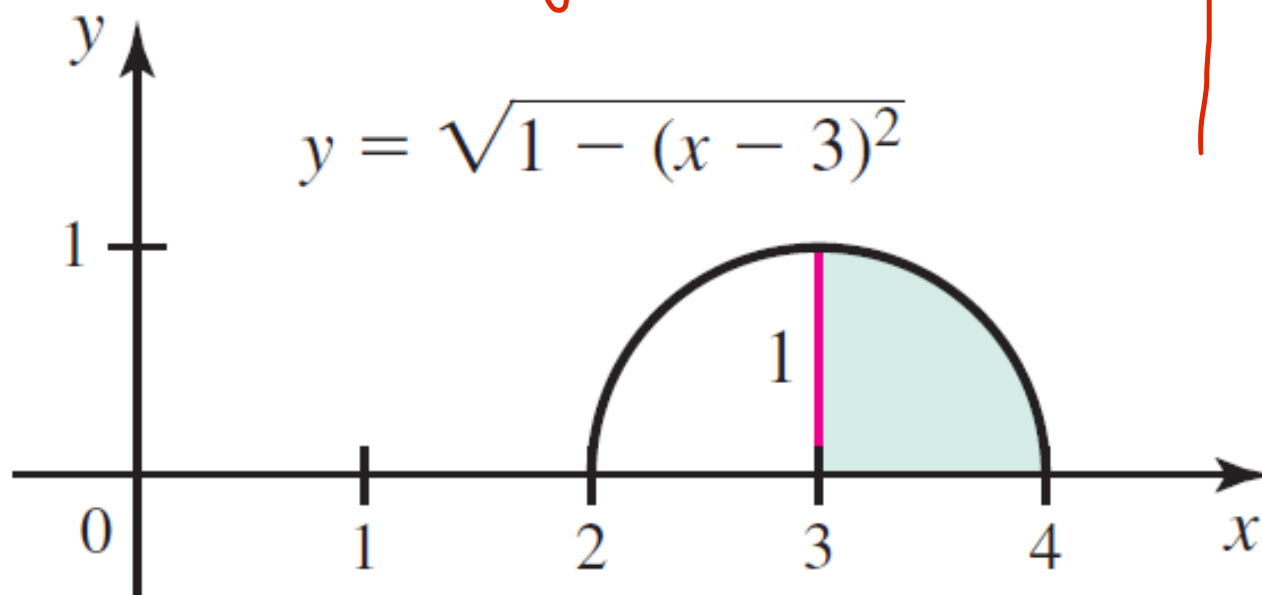
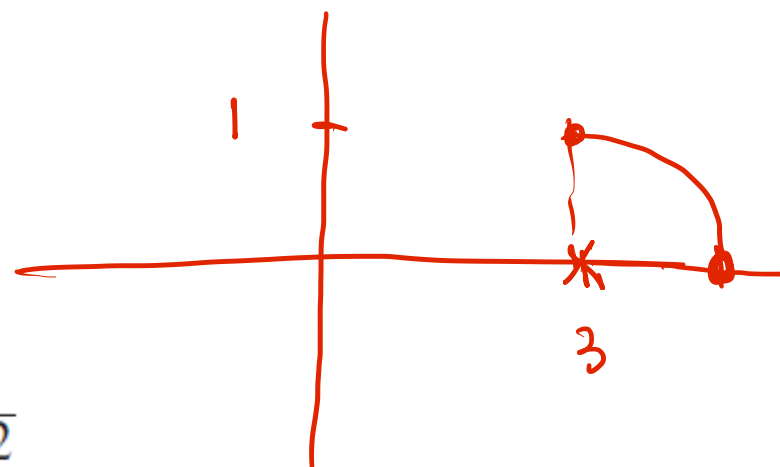
$$= \frac{1}{4} \pi \cdot 1^2 = \frac{1}{4} \pi$$

$$y = f(x) = \sqrt{1 - (x-3)^2} \text{ on } [3, 4]$$

Figure 5.27

$$y^2 = 1 - (x-3)^2$$

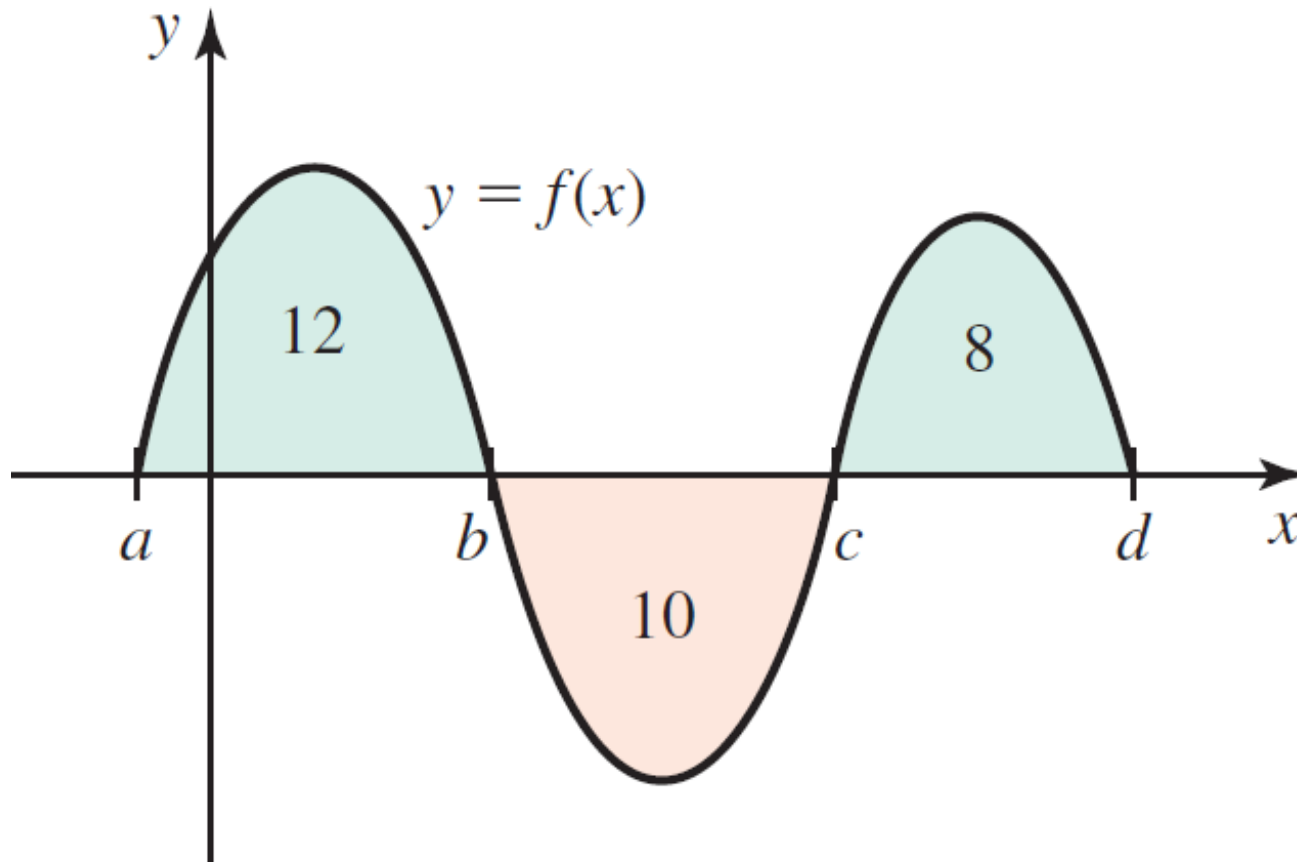
$$(x-3)^2 + y^2 = 1$$



$$\text{Area of shaded region} = \frac{1}{4} \pi (1)^2 = \frac{1}{4} \pi$$

- evaluate integrals from graph. $\int_a^d f(x) dx = \int_a^b f + \int_b^c f + \int_c^d f = 12 - 10 + 8 = 10$.

Figure 5.28



$$\int_a^b f(x) dx = \lim_{\Delta x \rightarrow 0} R_n, \quad R_n = \underbrace{f(x_1^*)}_{\text{red circle}} \Delta x + \dots + f(x_n^*) \Delta x$$

Table 5.4 Properties of Definite Integrals

Let f and g be integrable functions on an interval that contains a , b , and p .

1. $\int_a^a f(x) dx = \underline{0}$ Definition

2. $\int_a^b f(x) dx = - \int_b^a f(x) dx$ Definition

3. $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

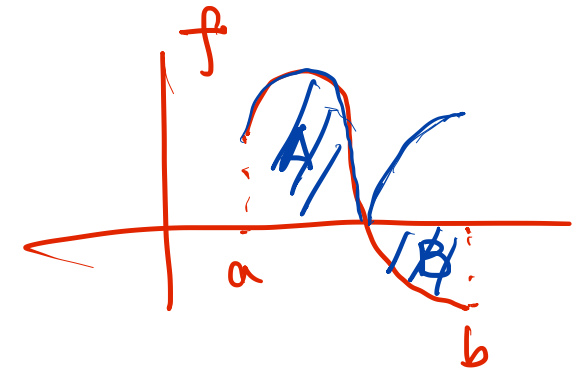
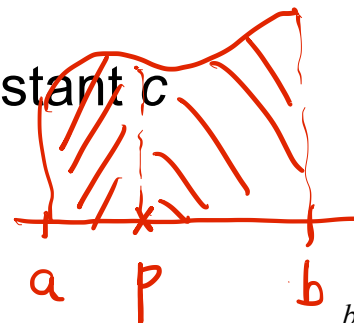
4. $\int_a^b cf(x) dx = c \int_a^b f(x) dx$, for any constant c

5. $\int_a^b f(x) dx = \int_a^p f(x) dx + \int_p^b f(x) dx$

6. The function $|f|$ is integrable on $[a, b]$, and $\int_a^b |f(x)| dx$ is the sum of the areas of the regions bounded by the graph of f and the x -axis on $[a, b]$.



$$\int_a^b f dx = A - B \quad || \int$$



$$\int_a^b |f(x)| dx \quad || \quad A+B$$

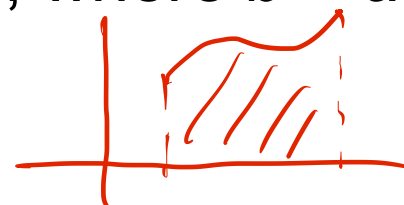
Table 5.5 Additional Properties of Definite Integrals

$$\underline{f(1)=a}, \underline{f(2)=b}$$

$$\frac{a+b}{2}$$

Let f and g be integrable functions on $[a, b]$, where $b > a$.

7. If $\underline{f(x) \geq 0}$ on $\underline{[a, b]}$, then $\int_a^b f(x) dx \geq 0$.



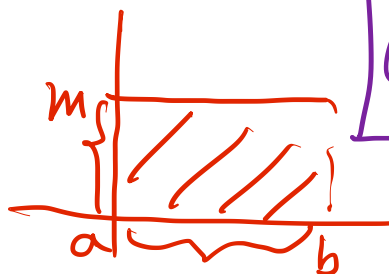
8. If $\underline{f(x) \geq g(x)}$ on $\underline{[a, b]}$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$. $\Leftrightarrow \int_a^b (f - g) dx \geq 0$

9. If $\underline{m} \leq \underline{f(x)} \leq \underline{M}$, then $\underline{m(b-a)} \leq \int_a^b f(x) dx \leq \underline{M(b-a)}$.

$$\int_a^b m dx = m(b-a)$$

$$\int f'(x) dx = f(x) + C$$

$$\frac{d}{dx} \int f(x) dx = f(x)$$



$$m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M$$

§5.3 Fundamental Theorem of Calculus

(a connection between differential calculus and integral calculus)

• Fundamental Theorem of Calculus, Part I

Assume that f is continuous on $[a, b]$

$\Rightarrow$ (1) $\underline{g(x) = \int_a^x f(t) dt}$ is continuous on $[a, b]$ and differentiable on (a, b)

(2) $\underline{g'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)}$.



Proof

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right]$$

$= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt = f(x)$

On $[x, x+h]$ $h f(u) \leq \int_x^{x+h} f(t) dt \leq h f(v)$, $u, v \in [x, x+h]$

$$f(x) \leftarrow f(u) \leq \frac{1}{h} \int_x^{x+h} f(t) dt \leq f(v) \rightarrow f(x)$$

Examples

(1) Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2} dt$

$$g'(x) = \sqrt{1+x^2}$$

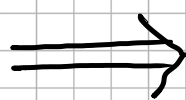
$$g(x) = \int_1^{\textcircled{x}} \sec t dt$$

$$(2) \text{ Find } \frac{d}{dx} \int_1^{x^4} \sec t dt = \frac{d}{dx} \int_1^{\textcircled{y(x)}=x^4} \sec t dt = \frac{d}{dx} g(y(x))$$
$$= g'(y(x)) y'(x) = \sec x^4 \cdot 4x^3$$

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt$$

• Fundamental Theorem of Calculus, Part II

Assume that f is continuous on $[a, b]$
and that $\underline{F'(x) = f(x)}$



$$\boxed{\int_a^b f(x) dx = F(b) - F(a)}$$
$$= F(x) \Big|_a^b$$

Proof $g(x) = \int_a^x f(t) dt, \quad \underline{g'(x) = f(x)}$

$$(F-g)' = F' - g' = f - f = 0 \implies F(x) - g(x) = C$$

$$\underline{F(x) - \int_a^x f(t) dt = C}$$

$$\underline{x=b}$$

$$F(b) - \int_a^b f(t) dt = C$$

$$\int_a^b f(t) dt = F(b) - C$$

$$\underline{x=a} \quad C = F(a) - \int_a^a f(t) dt = F(a)$$

Examples

(1) Evaluate the integral $\int_1^3 e^x dx = e^x \Big|_1^3 = e^3 - e^1$

(2) Find the area under the parabola $y = x^2$ from 0 to 1.

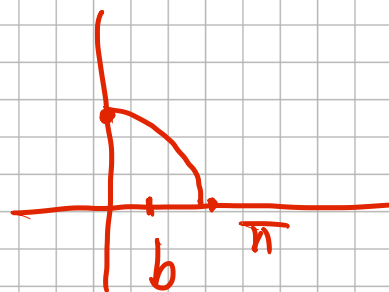
$$\int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$$



(3) Evaluate $\int_3^6 \frac{1}{x} dx = \ln|x| \Big|_3^6 = \ln 6 - \ln 3 = \ln 2$

(4) Find the area under the cosine curve from 0 to $b \in [0, \frac{\pi}{2}]$.

$$\int_0^b \cos x dx = \sin x \Big|_0^b = \sin b$$



(5) What is wrong with the following calculation?

$$\int_{-1}^3 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_{x=-1}^{x=3} = - \left[\frac{1}{3} + 1 \right] = -\frac{4}{3}.$$

$$\left(-\frac{1}{x} \right)' \neq \frac{1}{x^2}$$

$x=0$

$$\int_{-1}^3 \frac{1}{x^2} dx = \int_{-1}^0 + \int_0^3 \frac{1}{x^2}$$

- differentiation and integration as inverse processes

Assume that f is continuous on $[a, b]$

$$\Rightarrow (1) \quad \frac{d}{dx} \int_a^x f(t) dt = \underline{f(x)}$$

$$(2) \quad \int_a^b \frac{d}{dx} F(x) dx = F(b) - F(a) = \underline{F(x)} \Big|_a^b$$

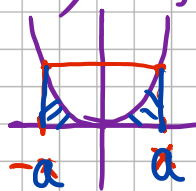
§5.4 Working with Integrals

integrating even and odd functions

- $f(x)$ is an even function $\iff f(-x) = f(x)$

$$\implies \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

$$f(x) = x^2 \quad f(-x) = (-x)^2 = x^2 = f(x)$$



$$f(x) = |x|$$

sym. w.r.t. y-axis

$$f(x) = \cos x$$



$$f(x) = x$$

$$f(-x) = -x = -f(x)$$

- $f(x)$ is an odd function $\iff$

$$\underline{f(-x) = -f(x)}$$

sym. w.r.t. origin

$$\implies \int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx = 0$$

$$f(0) = 0 \iff f(0) = -f(0)$$

$$-\int_0^a f(x) dx$$

Examples

$$(1) \int_{-2}^2 (x^4 - 3x^3) dx = \int_{-2}^2 x^4 dx - 3 \int_{-2}^2 x^3 dx$$

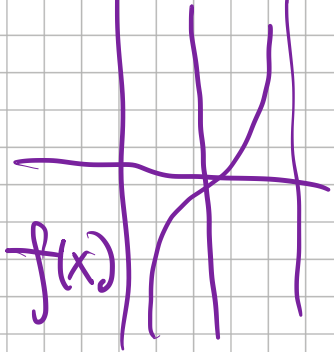
$$= 2 \int_0^2 x^4 dx = 2 \cdot \frac{1}{5} x^5 \Big|_0^2 = \frac{2}{5} \underline{\underline{2^5}}$$

$$(2) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x - 4 \sin^3 x) dx = 2 \int_0^{\frac{\pi}{2}} \cos x dx = 2 \sin x \Big|_0^{\frac{\pi}{2}} = 2 [1 - 0] = 2$$

$$[\sin(-x)]^3 = [-\sin x]^3 = -\sin^3 x$$

$$13) \int_{-1}^1 \frac{\tan x}{1+x^2+x^4} dx = 0$$

$$f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4} = \frac{\frac{\sin(-x)}{\cos(-x)}}{1+x^2+x^4} = \frac{-\sin x}{\cos x} = -f(x)$$



• average value of a function

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx$$

Example (average elevation) A hiking trail has an elevation given by

$$f(x) = \underline{60x^3} - \underline{650x^2} + \underline{1200x} + 4500,$$

where f is measured in feet above sea level and x represents horizontal distance along the trail in miles, with $0 \leq x \leq 5$.

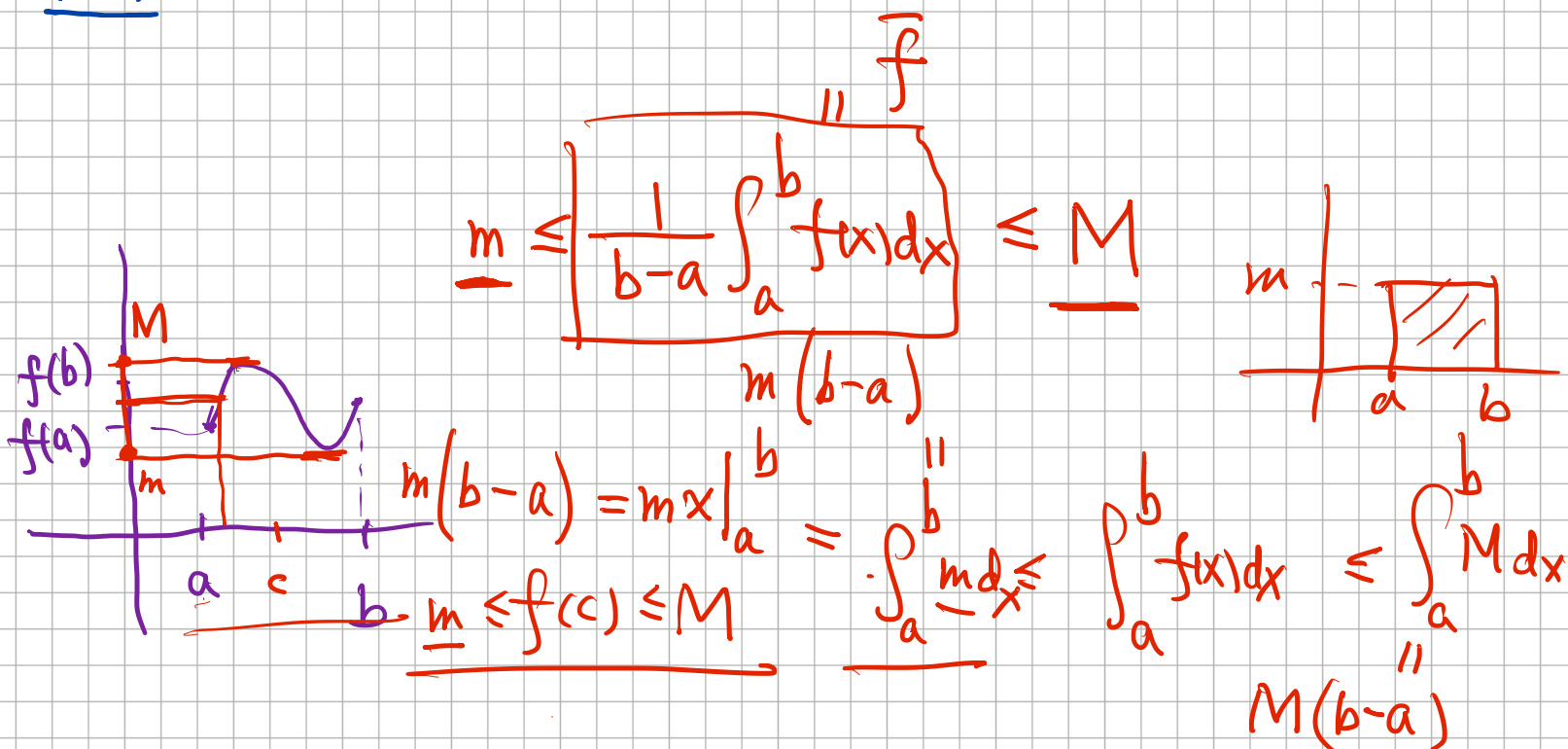
What is the average elevation of the trail?

$$\bar{f} = \frac{1}{5-0} \int_0^5 f(x) dx = \frac{1}{5} \left[\frac{60}{4} x^4 - \frac{650}{3} x^3 + \frac{1200}{2} x^2 + 4500x \right]_0^5$$

• mean value Theorem for integrals

f is continuous on $[a, b] \Rightarrow \exists \underline{c} \in (a, b)$, s.t. $\underline{f(c) = \frac{1}{b-a} \int_a^b f(x) dx}$

Proof



Example Find points on $(0, 1)$ at which $f(x) = 2x(1-x)$ equals its average value on $[0, 1]$.

$$\begin{aligned} \bar{f} &= \frac{1}{1-0} \int_0^1 2x(1-x) dx = 2 \int_0^1 (x - x^2) dx = 2 \left[\frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_0^1 \\ &= 2 \left[\frac{1}{2} - \frac{1}{3} \right] = 2 \cdot \frac{3-2}{2 \cdot 3} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \frac{1}{3} &= \bar{f} = f(c) = 2c(1-c) = 2(c - c^2) \Rightarrow c - c^2 = \frac{1}{6} \\ 0 &= c^2 - c + \frac{1}{6} \end{aligned}$$

- mean value Theorem for integrals

f is continuous on $[a, b] \implies \exists c \in (a, b)$, s.t. $f(c) = \frac{1}{b-a} \int_a^b f(x) dx$

Proof

Example Find points on $(0, 1)$ at which $f(x) = 2x(1-x)$ equals its average value on $[0, 1]$.

$$F'(x) = f(x)$$

$$\frac{d F(g(x))}{dx} = F'(g(x)) g'(x)$$

$$\int \frac{d}{dx} F(g(x)) dx = \int \underline{F'(g(x))} g'(x) dx = \int \underline{f(g(x)) g'(x)} dx$$

||

$$\underline{F(g(x))}$$

§5.5 Substitution Rule

$$\int 2x \sqrt{1+x^2} dx$$

$u = 1+x^2$
 $du = (1+x^2)' dx = 2x dx$
 $(u)^{\frac{1}{2}} = 2x dx$

$$\int \sqrt{u} du = \frac{1}{\frac{1}{2}+1} u^{\frac{1}{2}+1} + C$$

$$= \frac{2}{3} u^{\frac{3}{2}} + C$$

• substitution rule for indefinite integral

(1) $u = g(x)$ is differentiable on an interval
 and (2) f is continuous on Range(f)

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

Examples

$$(1) \int x^3 \cos(x^4+2) dx$$

$u = x^4+2$
 $du = 4x^3 dx$
 $u' = 4x^3$
 $x^3 dx = \frac{1}{4} du$

$$\int \frac{1}{4} \cos u du = \frac{1}{4} \sin u + C$$

$$= \frac{1}{4} \sin(x^4+2) + C$$

$$(2) \int \sqrt{2x+1} dx$$

$u = 2x+1$
 $du = 2 dx$
 $\Rightarrow dx = \frac{1}{2} du$

$$\int \frac{1}{2} \sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3} u^{\frac{1}{2}+1} + C$$

$$= \frac{1}{3} (2x+1)^{\frac{3}{2}} + C$$

$$\int x^2 \sqrt{2x^3+1} dx$$

$u = 2x^3+1$
 $du = 6x^2 dx$
 $x^2 dx = \frac{1}{6} du$

$$\int \frac{1}{6} \sqrt{u} du =$$

$$(3) \int \frac{x}{\sqrt{1-4x^2}} dx \quad \begin{array}{l} u=1-4x^2 \\ du=-8x dx \\ x dx = -\frac{1}{8} du \end{array} \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{8}\right) du$$

$$= -\frac{1}{8} \int u^{-\frac{1}{2}} du = -\frac{1}{8} \cdot \underline{\underline{2 \cdot u^{-\frac{1}{2}+1}}} + C = -\frac{1}{4} (1-4x^2)^{\frac{1}{2}} + C$$

$$(4) \int e^{5x} dx \quad \begin{array}{l} u=5x \\ du=5 dx \\ dx=\frac{1}{5} du \end{array} \int e^u \cdot \frac{1}{5} du = \frac{1}{5} e^u + C$$

$$= \frac{1}{5} e^{5x} + C$$

$$\rightarrow x^2 = u-1 \Rightarrow x^4 = (u-1)^2$$

$$(5) \int x^5 \sqrt{1+x^2} dx \quad \begin{array}{l} \boxed{u=1+x^2} \\ du=2x dx \\ x dx = \frac{1}{2} du \end{array} \int \underline{x^4} \sqrt{u} \cdot \frac{1}{2} du$$

$$x^4: \underline{\underline{x dx}}$$

$$\int (u-1) \sqrt{u} du = \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) du$$

$$= \frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \frac{1}{2} \int (u-1)^2 \sqrt{u} du$$

$$= \frac{1}{2} \int (u^2 - 2u + 1) \sqrt{u} du$$

$$= \frac{1}{2} \int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}) du$$

$$= \frac{1}{2} \left[\frac{2}{7} u^{\frac{7}{2}} - \frac{4}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{\frac{3}{2}} \right] + C$$

$$(6) \int \tan x dx$$

$$= \int \frac{\sin x}{\cos x} dx$$

$$\begin{array}{l} u = \cos x \\ du = -\sin x dx \\ \sin x dx = -du \end{array} \int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$\begin{array}{l} u = \sin x \\ du = \cos x dx \\ dx = \frac{1}{\cos x} du \end{array}$$

$$\int \frac{u}{\cos x} \cdot \frac{1}{\cos x} du = \int \frac{u}{\cos^2 x} du$$

$$= \int \frac{u}{1-u^2} du \quad \begin{array}{l} v=1-u^2 \\ dv=-2u du \\ u du = -\frac{1}{2} dv \end{array} -\frac{1}{2} \int \frac{1}{v} dv$$

$$= -\frac{1}{2} \ln|1-\sin^2 x| + C = -\frac{1}{2} \ln|\cos^2 x| + C$$

• substitution rule for definite integral

$$\int_a^b f(g(x))g'(x)dx \xrightarrow{u=g(x)} \int_{g(a)}^{g(b)} f(u)du$$

Proof

$$(7) \int_0^4 \sqrt{2x+1} dx \quad \begin{array}{l} u=2x+1: 1 \rightarrow 9 \\ du=2dx \\ dx=\frac{1}{2}du \end{array} \quad \begin{array}{l} \frac{1}{2} \int_1^9 u^{\frac{1}{2}} du = \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^9 \\ = \frac{1}{3} \left[(3^2)^{\frac{3}{2}} - 1 \right] = \frac{1}{3} [27 - 1] \end{array}$$

$$(8) \int_1^2 \frac{1}{(3-5x)^2} dx \quad \begin{array}{l} u=3-5x: -2 \rightarrow -7 \\ du=-5dx \\ dx=-\frac{1}{5}du \end{array} \quad \begin{array}{l} \int_{-2}^{-7} u^{-2} \left(-\frac{1}{5}\right) du \\ = -\frac{1}{5} (-1) u^{-1} \Big|_{-2}^{-7} = \frac{1}{5} \left[\frac{1}{-7} - \frac{1}{-2} \right] = -\frac{1}{5} \left[\frac{1}{2} - \frac{1}{7} \right] \\ = -\frac{1}{5} \cdot \frac{5}{14} = -\frac{1}{14} \end{array}$$

$$(9) \int_1^e \frac{\ln x}{x} dx \quad \begin{array}{l} u=\ln x: 0 \rightarrow 1 \\ du=\frac{1}{x} dx \end{array} \quad \int_0^1 u du = \frac{1}{2} u^2 \Big|_0^1 = \frac{1}{2}$$

$$(10) \int_{-2}^2 (x^6 + 1) dx = 2 \int_0^2 (x^6 + 1) dx = 2 \left[\frac{1}{7} x^7 + x \right]_0^2$$

$$(11) \int_{-1}^1 \frac{\tan x}{1 + x^2 + x^4} dx = 0 \quad ?$$

$\sin x \rightarrow$ odd
 $\cos x \rightarrow$ even

Table 5.6 General Integration Formulas

$$u = ax \quad du = a dx \Rightarrow dx = \frac{1}{a} du$$

$$1. \int \cos ax \, dx = \frac{1}{a} \sin ax + C$$

$$\int \cos u \cdot \frac{1}{a} du = \frac{1}{a} \sin(ax) + C$$

$$3. \int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C$$

$$2. \int \sin ax \, dx = -\frac{1}{a} \cos ax + C$$

$$4. \int \csc^2 ax \, dx = -\frac{1}{a} \cot ax + C$$

$$5. \int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C$$

$$6. \int \csc ax \cot ax \, dx = -\frac{1}{a} \csc ax + C$$

$$7. \int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$$

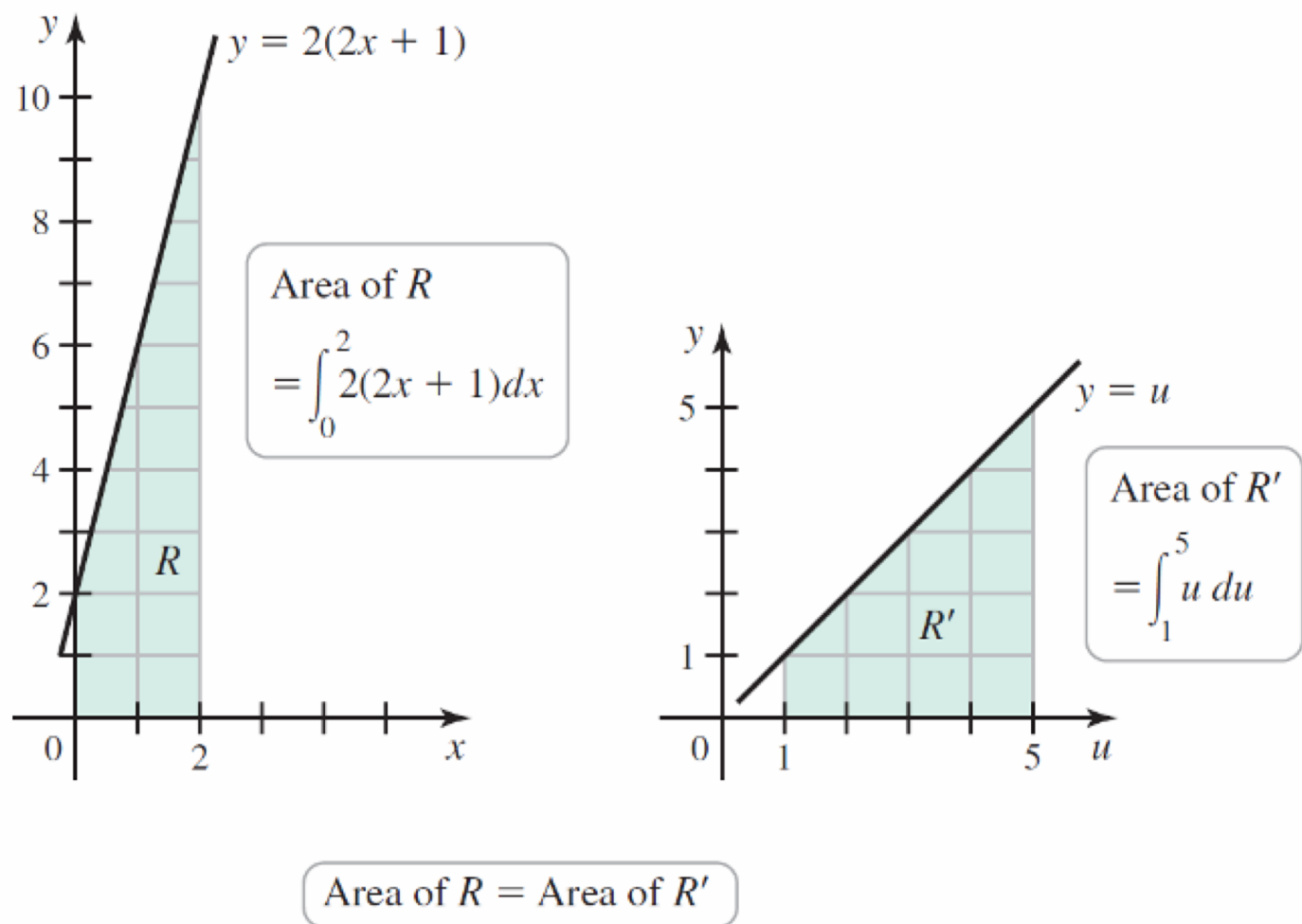
$$8. \int b^x \, dx = \frac{1}{\ln b} b^x + C, b > 0, b \neq 1$$

$$9. \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$10. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C, a > 0$$

$$11. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C, a > 0$$

Figure 5.58



§7.2 Exponential Models

Let $y(t)$ be the value of a quantity y at time t

$$\frac{dy(t)}{dt} = k y(t)$$

- law of natural growth if $k > 0$
- law of natural decay if $k < 0$

ordinary differential equation

$$y(t) = \underbrace{y_0}_{\text{initial value}} e^{\underbrace{k}_{\text{growth/decay rate}} t}$$

$$\int \frac{1}{y} dy = \int k dt$$

$$y(0) = P e^0 = P$$

$$e^{\ln|y|} = e^{kt+C}$$

$$|y| = e^C e^{kt}$$

$$y = (\pm e^C) e^{kt} = \underline{D} e^{kt}$$

$$2P_0 = P(T_2) = P_0 e^{kT_2}$$

$$\Rightarrow 2 = e^{kT_2} \Rightarrow \ln 2 = kT_2$$

- population growth $P(t)$ denotes the size of a population at time t

$$\frac{dP}{dt} = kP \text{ or } k = \frac{dP}{dt} / P$$

$$\Rightarrow P(t) = \underline{P_0} e^{kt}$$

doubling time

$$\underline{T_2} = \frac{\ln 2}{k}$$

- radioactive decay $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t

$$\frac{dm}{dt} = km \text{ with } k < 0 \Rightarrow m(t) = m_0 e^{kt}$$

$$\frac{1}{2} m_0 = m(T_{1/2}) = m_0 e^{kT_{1/2}}$$

$$\frac{1}{2} = e^{kT_{1/2}} \Rightarrow \ln\left(\frac{1}{2}\right) = kT_{1/2}$$

half-life

$$\underline{T_{1/2}} = \frac{\ln(1/2)}{k}$$

Example 2 (World population) Human population growth rates vary geographically and fluctuate over time. The overall growth rate for world population peaked at an annual rate of 2.1% per year in the 1960s. Assume a world population of 6 billion in 1990 ($t=0$) and 7.4 billion in 2017 ($t=18$).

(a) Find an exponential growth function that fits the two data pts.

(b) Find the doubling time using the model in (a).

(c) Find the (absolute) growth rate $P'(t)$.

(d) How fast is the population expected to be growing in 2020 ($t=21$)?

$$(a) \quad P(t) = P_0 e^{kt} \quad P_0 = 6, \quad k = ?$$

$$P(t) = 6e^{kt} \quad 7.4 = P(18) = 6e^{18k} \Rightarrow \frac{7.4}{6} = e^{18k}$$

$$\ln \frac{7.4}{6} = 18k \Rightarrow k = \ln \frac{7.4}{6} / 18 \approx 0.01165 \text{ b/y.}$$

$$\boxed{P(t) = 6e^{0.01165t}}$$

$$(b) \quad T_2 = \ln(2) / 0.01165 \approx 59.5 \text{ y.}$$

$$(c) \quad P'(t) = 6 \times (0.01165) e^{0.01165t} = 0.0699 e^{0.01165t}$$

$$(d) \quad P'(21) = 0.0699 e^{0.01165 \times 21} = 89.3 \text{ mil/y}$$

Example 5 Researchers determine that a fossilized bone has 30% of the C-14 of a live bone. Estimate the age of the bone. Assume a half-life for C-14 of 5730 years.

$$m(t) = m_0 e^{kt}$$

$$T_{1/2} = 5730 \text{ years.}$$

$$\frac{1}{2} m_0 = m(T_{1/2}) = m_0 e^{k T_{1/2}}$$

$$\Rightarrow T_{1/2} = \frac{\ln(1/2)}{k} \Leftrightarrow k = \frac{\ln(1/2)}{T_{1/2}}$$

$$k = \frac{\ln(1/2)}{5730} \approx (-) 0.000121 \text{ /year}$$

$$\underline{0.3 m_0} = m(T^*) = m_0 e^{-0.000121 T^*}$$

$$\ln(0.3) = -0.000121 T^*$$

$$T^* = - \frac{\ln(0.3)}{0.000121} \approx 9950 \text{ years}$$

Example 6 An exponential decay function $y(t) = y_0 e^{kt}$ models the amount of drug in the blood t hr after an initial dose of $y_0 = 100 \text{ mg}$ is administered. Assume the half-life of the drug is 16 hours.

(a) Find the exponential function that governs the amount of drug in the blood.

(b) How much time is required for the drug to reach 1% of the initial dose?

(c) If a second 100-mg dose is given 12 hr. after the first dose, how much time is required for the drug level to reach 1 mg?

$$(a) y(t) = 100 e^{kt} \quad k = ? \quad \ln\left(\frac{1}{2}\right) / T_{1/2} = \ln\left(\frac{1}{2}\right) / 16$$

$$y(t) = 100 e^{-0.043t} \quad \approx -0.0433 \text{ per hr.}$$

$$(b) 0.01 \times 100 = 100 e^{-0.043t} \Rightarrow t =$$

$$t = \frac{\ln(0.01)}{k} = \frac{\ln(0.01)}{-0.0433} \approx 106 \text{ hr.}$$

$$(c) y(12) = 100 e^{-0.043 \times 12} \approx 59.5 \text{ mg.}$$

$$y(t) = 159.5 e^{-0.043t} \quad 1 = y(T) = 159.5 e^{-0.043T}$$

$$e^{-0.043T} = \frac{1}{159.5}$$

$$T = \ln\left(\frac{1}{159.5}\right) / -0.043 \approx 117.1$$